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## Conway's groupoid and its relatives

Nick Gill, Neil I. Gillespie, Cheryl E. Praeger, and Jason Semeraro

*Dedicated to John Horton Conway on the occasion of his 75<sup>th</sup> birthday*

**ABSTRACT.** In 1987, John Horton Conway constructed a subset  $M_{13}$  of permutations on a set of size 13 for which the subset fixing any given point is isomorphic to the Mathieu group  $M_{12}$ . The construction has fascinated mathematicians for the past thirty years, and remains remarkable in its mathematical isolation. It is based on a “moving-counter puzzle” on the projective plane  $\text{PG}(2, 3)$ . This survey, an homage to John Conway and his mathematics, discusses consequences and generalisations of Conway's construction. In particular it explores how various designs and hypergraphs can be used instead of  $\text{PG}(2, 3)$  to obtain interesting analogues of  $M_{13}$ . In honour of John Conway, we refer to these analogues as *Conway groupoids*. A number of open questions are presented.

### 1. The first Conway groupoid $M_{13}$

In 1987, John Conway published the first of two celebrated papers [12, 13] in which he described his construction of the sporadic simple group of Mathieu,  $M_{12}$ , via a “moving-counter puzzle” on the projective plane  $\text{PG}(2, 3)$  of order 3. Conway had noticed some new structural links between two permutation groups, namely  $M_{12}$ , which acts 5-transitively on 12 letters, and  $\text{PSL}_3(3)$ , which acts 2-transitively on the 13 points of  $\text{PG}(2, 3)$ . These led him to his construction of  $M_{13}$ . In [13, page 1] he writes:

To be more precise, the point-stabilizer in  $\text{PSL}_3(3)$  is a group of structure  $3^2 : 2\text{Sym}(4)$  that permutes the 12 remaining points imprimitively in four blocks of 4, and there is an isomorphic subgroup of  $M_{12}$  that permutes the 12 letters in precisely the same fashion. Again, the line-stabilizer in  $\text{PSL}_3(3)$  is a group of this same structure, that permutes the 9 points not on that line in a doubly transitive manner, while the stabilizer of a triple in  $M_{12}$  is an isomorphic group that permutes the 9 letters not in that triple in just the same manner.

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In the heady days when new simple groups were being discovered right and left, this common structure suggested that there should be a new group that contained both  $M_{12}$  and  $\text{PSL}_3(3)$ , various copies of which would intersect in the subgroups mentioned above.

The putative “new group” does not of course exist, but Conway’s construction of  $M_{12}$  using a certain puzzle on  $\text{PG}(2, 3)$  did yield a natural definition of a *subset* of permutations that contains both of these groups in the manner just described, and Conway called it  $M_{13}$ .

There has been a great deal of work devoted to the study of generating simple groups in combinatorial ways (Curtis’ book [15] gives a comprehensive account of one aspect of this study). Despite this, Conway’s construction of  $M_{12}$  seems quite distinct from these, and remains somewhat mysterious.

Our aim in this survey is, first, to discuss Conway’s original construction and its principal properties, as well as some of the variants that were first considered in depth in [14]. We then cast our net a little wider, and discuss more recent and more broad-ranging generalizations that involve “puzzles” on other structures including families of designs, regular two graphs, 4-uniform hypergraphs, and the like. As with Conway’s original construction, these have led to unexpected bonuses, such as new error correcting codes. We hope that the results we survey, and the questions they raise, will provide data for future investigations which may in turn shed new light on Conway’s beautiful construction.

**1.1. Conway’s original puzzle.** We now present Conway’s original puzzle in terms of permutations, rather than the “counters” used in [13, Section 2]. This description bears little resemblance to what we traditionally think of as a “puzzle”, but contains all of the salient mathematics. Our notation, too, is different from that of Conway but prepares the way for what will come later.

We write  $\Omega$  for the set of 13 points of  $\text{PG}(2, 3)$ . Each of the 13 lines of  $\text{PG}(2, 3)$  is incident with exactly 4 points, and each pair of points is incident with exactly one line. We think of a line as simply a 4-subset of  $\Omega$ . Then, given any pair of distinct points  $a, b \in \Omega$  we define the *elementary move*, denoted  $[a, b]$  to be the permutation  $(a, b)(c, d)$  where  $\{a, b, c, d\}$  is the unique line in  $\text{PG}(2, 3)$  containing  $a$  and  $b$ . For a point  $a \in \Omega$ , it is convenient to define the *move*  $[a, a]$  to be the identity permutation. Then, given a sequence of (not necessarily distinct) points  $a_1, a_2, \dots, a_\ell$  we define the *move*

$$(1.1) \quad [a_1, a_2, \dots, a_\ell] = [a_1, a_2] \cdot [a_2, a_3] \cdots [a_{\ell-1}, a_\ell].$$

Note that we apply  $[a_1, a_2]$  first, and then  $[a_2, a_3]$ , and so on, so this move maps  $a_1$  to  $a_\ell$ .<sup>1</sup> Observe that all of these moves are elements of  $\text{Sym}(13)$ .

We now choose a point of  $\text{PG}(2, 3)$ , label it  $\infty$ , and consider two subsets of  $\text{Sym}(13)$ :

$$(1.2) \quad \begin{aligned} \pi_\infty(\text{PG}(2, 3)) &:= \{[a_1, a_2, \dots, a_\ell] \mid 1 < \ell \in \mathbb{Z}, a_1, \dots, a_\ell \in \Omega, a_1 = a_\ell = \infty\}; \\ \mathcal{L}_\infty(\text{PG}(2, 3)) &:= \{[a_1, a_2, \dots, a_\ell] \mid 1 < \ell \in \mathbb{Z}, a_1, \dots, a_\ell \in \Omega, a_1 = \infty\}. \end{aligned}$$

<sup>1</sup>The terminology intentionally suggests a “puzzle” in which, for example, the move  $[\infty, a, b]$  denotes “moving” a “counter” from  $\infty$  first to  $a$ , using the elementary move  $[\infty, a]$ , and then moving it from  $a$  to  $b$ , using the elementary move  $[a, b]$ , and so on.

It is easy to see that the set  $\pi_\infty(\text{PG}(2, 3))$  (which we call the *hole-stabilizer*) is, in fact, a subgroup of  $\text{Sym}(\Omega \setminus \{\infty\}) \cong \text{Sym}(12)$ . Much less trivial is the spectacular fact due to Conway that  $\pi_\infty(\text{PG}(2, 3))$  is isomorphic to  $M_{12}$  ([13, Sections 3 and 7]). The set  $\mathcal{L}_\infty(\text{PG}(2, 3))$  is a subset of  $\text{Sym}(13)$  that contains  $M_{12}$ . It has size  $13 \cdot |M_{12}|$  and is equal to the product  $\pi_\infty(\text{PG}(2, 3)) \cdot \text{Aut}(\text{PG}(2, 3))$ . Since  $\text{Aut}(\text{PG}(2, 3)) \cong \text{PSL}(3, 3)$ , the set  $M_{13}$  contains both  $M_{12}$  and  $\text{PSL}(3, 3)$  and, moreover, contains copies of these groups intersecting in exactly the manner that Conway proposed in the quote above.

**1.2. Some variants.** A number of variants of the  $M_{13}$ -puzzle are mentioned by Conway in his original article [13]. Other variants were investigated further by Conway, Elkies and Martin [14], two of which relate to  $\text{PG}(2, 3)$  and are particularly interesting:

- (1) **The signed game.** One defines moves on  $\text{PG}(2, 3)$  as before, except that the definition of an elementary move also assigns a “sign” to each letter in the permutation. We write  $[a, b] = (a, b)(\underline{c}, \underline{d})$  to denote that the letters  $c$  and  $d$  are given negative signs. The resulting hole-stabilizer,  $\pi_\infty(\text{PG}(2, 3))$ , may be regarded as a subgroup of the wreath product  $(\mathbb{Z}/2\mathbb{Z}) \wr \text{Sym}(12)$  and turns out to be isomorphic to  $2M_{12}$ , the double-cover of  $M_{12}$ , [14, Theorem 3.5].
- (2) **The dualized game.** In this puzzle, the set  $\Omega$  is the union of the point-set and the line-set of  $\text{PG}(2, 3)$ . We distinguish both a point  $\infty$ , and a line  $\overline{\infty}$  such that  $\infty$  and  $\overline{\infty}$  are incident in  $\text{PG}(2, 3)$ . Since  $\text{PG}(2, 3)$  is self-dual, one can define moves, as in the original game, for sequences of points as well as sequences of lines. Once one has done this, one can define a move of the form

$$[p_1, q_1, p_2, q_2, \dots, p_\ell, q_\ell] := [p_1, \dots, p_\ell] \cdot [q_1, \dots, q_\ell]$$

where  $p_1, \dots, p_\ell$  are points  $q_1, \dots, q_\ell$  are lines and we require that  $q_i$  is incident with  $p_i$  for all  $i = 1, \dots, \ell$ , and  $q_i$  is incident with  $p_{i+1}$  for all  $i = 1, \dots, \ell-1$ . One can define analogously a hole-stabilizer  $\pi_\infty^d(\text{PG}(2, 3))$ , except that here we require  $p_1 = p_\ell = \infty$  and  $q_1 = q_\ell = \overline{\infty}$ . The group  $\pi_\infty^d(\text{PG}(2, 3))$  is isomorphic to  $M_{12}$  and its action on  $\Omega$  splits into two orbits: the point-set and line-set. By interchanging the point-set and line-set appropriately one can obtain a concrete representation of the outer automorphism of  $M_{12}$ , [14, Section 4].

**1.3. Sharp multiple transitivity.** A subset  $K$  of  $\text{Sym}(n)$  is called *sharply  $t$ -transitive* if, for any pair of  $t$ -tuples of distinct elements from  $\{1, \dots, n\}$ , there is exactly one element of  $K$  mapping the first  $t$ -tuple to the second. A classical result of Jordan asserts that there are no sharply  $t$ -transitive subgroups of  $\text{Sym}(n)$  for  $6 \leq t < n-2$ ; moreover the group  $M_{12}$  is the only finite permutation group which is sharply 5-transitive on a set of size  $n > 7$  (see, for instance [37]).

Thus, if one seeks a sharply 6-transitive subset of some finite symmetric group, one must consider subsets that are not groups. The set  $M_{13}$ , having size  $13 \cdot |M_{12}|$  and containing a sharply 5-transitive permutation group, appears to be a good candidate to be a sharply 6-transitive subset of  $\text{Sym}(13)$ . However Bonisoli and Quattrocchi have proved that this is not the case (as well as much more) [6].

Despite this the set  $M_{13}$  does appear to possess strong transitivity properties and in order to study these properties, Conway, Elkies and Martin introduced the following notions in [14, Section 5.1]. Here  $\mathcal{P}$  denotes the set of all ordered 6-tuples of distinct points of  $\text{PG}(2, 3)$ .

- A tuple  $\mathbf{p} \in \mathcal{P}$  is a *universal donor* if, for all  $\mathbf{q} \in \mathcal{P}$ , there exists  $g \in M_{13}$  such that  $\mathbf{p}^g = \mathbf{q}$ .
- A tuple  $\mathbf{q} \in \mathcal{P}$  is a *universal recipient* if, for all  $\mathbf{p} \in \mathcal{P}$ , there exists  $g \in M_{13}$  such that  $\mathbf{p}^g = \mathbf{q}$ .

They proved the following result [14, Theorems 5.2 and 5.3] which gives full information regarding transitivity properties of  $M_{13}$  on 6-tuples.

**THEOREM 1.1.** (1) *A tuple  $\mathbf{p} = (p_1, \dots, p_6) \in \mathcal{P}$  is a universal donor if and only if  $p_i = \infty$  for some  $i$ .*  
 (2) *A tuple  $\mathbf{q} \in \mathcal{P}$  is a universal recipient if and only if  $\mathbf{q}$  contains a line of  $\text{PG}(2, 3)$ .*

**1.4. Motivation for our survey.** In this survey we introduce Conway groupoids in a more general setting. Here we make a few general remarks about the surprising (to us) connections between Conway groupoids and other combinatorial areas. We hope that by reviewing these connections, and raising several open questions, others may be drawn to their study and perhaps explain more clearly the particular phenomenon of Conway's  $M_{13}$ .

We discuss how Conway groupoids have been used to classify a family of block designs (Section 4.2). Exploring the Conway groupoids for these designs revealed a link between them and the family of graphs with the ‘strong triangle property’. It turned out that this family of graphs had been well-studied because of their links with the Fischer 3-transposition groups. Moreover the sub-family of those graphs most closely connected with the 3-transposition groups had been classified and this classification, in turn, allowed the classification of the corresponding family of block designs. Moreover an alternative proof, using Fischer’s classification of 3-transposition groups, led to a novel description of the sets of 3-transpositions in these groups in terms of the elementary moves of Conway groupoids.

Reflecting on this example, one wonders whether other combinatorial objects can be classified using Conway groupoids. Perhaps certain families of regular two-graphs? It would be rather strange if the block designs/graphs in Section 4.2 were the only ones amenable to classification via these methods.

Reflecting further, it seems that interesting Conway groupoids built from block designs, where the hole-stabilizer (defined in Section 2) is primitive but does not contain the alternating group, are associated with interesting families of error correcting codes. In particular, study of some families of Conway groupoids has led to the discovery of new families of completely transitive codes, with good combinatorial and symmetry properties (Section 3). Might these examples exemplify a general pattern? For example, might we always find codes with nice properties (uniformly packed (in the wide sense), completely regular, or completely transitive) when we have interesting Conway groupoids? One possible strategy might be to examine whether techniques used by Assmus and Key [2] to study the  $p$ -ranks of incidence matrices for Steiner triple systems could be adapted for these supersimple block designs.

Conway groupoids related to other structures are explored in Section 5: ranging from regular two-graphs, to abstract groups, to the collinearly complete designs arising from the work of Higman and McLaughlin [24]. There is relatively little known about them, and the open questions posed in that section are just several of many one could ask.

## 2. A more general setting for groupoids

For the rest of this paper we turn our attention to work inspired by Conway's construction of  $M_{13}$ , and which seeks to generalize it in various ways. This more general setting was first considered in [18]; it involves the notion of a 4-hypergraph, namely a pair  $\mathcal{D} := (\Omega, \mathcal{B})$ , where  $\Omega$  is a finite set of size  $n$ , and  $\mathcal{B}$  is a finite multiset of subsets of  $\Omega$  (called lines), each of size 4. Observe that  $\text{PG}(2, 3)$  is a 4-hypergraph on a set of size 13. A pair  $a, b$  of (not necessarily distinct) points is called *collinear* if  $a, b$  are contained in some line of  $\mathcal{D}$ , and  $\mathcal{D}$  is said to be *connected* if, for all  $a, b \in \Omega$ , there exists a finite sequence  $a_0 = a, a_1, \dots, a_k = b$  of points from  $\Omega$  such that each pair  $a_{i-1}, a_i$  is collinear.

Consider an arbitrary connected 4-hypergraph  $\mathcal{D} = (\Omega, \mathcal{B})$ . For a pair of distinct collinear points  $a, b \in \Omega$  we define the *elementary move*, denoted  $[a, b]$ , to be the permutation

$$(a, b)(c_1, d_1)(c_2, d_2) \cdots (c_\lambda, d_\lambda)$$

where  $\{a, b, c_i, d_i\}$  (for  $i = 1, \dots, \lambda$ ) are the lines of  $\mathcal{D}$  containing  $a$  and  $b$ . The value of  $\lambda$  in general depends on  $a$  and  $b$ . To ensure that each elementary move is well defined, the 4-hypergraph  $\mathcal{D}$  is required to be *pliable*, that is, whenever two lines have at least three points in common, the two lines contain exactly the same points.

The rest of the set-up proceeds *à la* the analysis of  $M_{13}$  given at the start of §1.1: We define the move  $[a, a]$  to be the identity permutation, for each  $a$ , and for a sequence  $a_1, a_2, \dots, a_\ell$  such that each pair  $a_{i-1}, a_i$  is collinear, we define the move  $[a_1, \dots, a_\ell]$  exactly as in (1.1). Finally, we distinguish a point of  $\mathcal{D}$  which we call  $\infty$ , and we define the hole-stabilizer  $\pi_\infty(\mathcal{D})$  and the set  $\mathcal{L}_\infty(\mathcal{D})$  as in (1.2). The set  $\pi_\infty(\mathcal{D})$  is again a subgroup of  $\text{Sym}(\Omega \setminus \{\infty\})$ , and the subset  $\mathcal{L}_\infty(\mathcal{D})$  of  $\text{Sym}(\Omega)$  is an analogue of Conway's  $M_{13}$ .

In [13], Conway recognised that  $M_{13}$  could be endowed with the structure of a *groupoid* (that is, a small category in which all morphisms are isomorphisms). The set  $M_{13}$  is sometimes referred to as the *Mathieu groupoid*. We define an analogue of this notion in this more general setting, and explain the connection between the set and the category.

For a pliable, connected 4-hypergraph  $\mathcal{D} = (\Omega, \mathcal{B})$ , the *Conway groupoid*  $\mathcal{C}(\mathcal{D})$  is the small category whose object set is  $\Omega$ , such that, for  $a, b \in \Omega$ , the set  $\text{Mor}(a, b)$  of morphisms from  $a$  to  $b$  is precisely

$$\text{Mor}(a, b) := \{[a, a_1, \dots, a_{k-1}, b] \mid a_{i-1}, a_i \in \Omega \text{ for } 1 \leq i \leq k-1\}.$$

Since  $\mathcal{D}$  is connected, there exists a finite sequence  $\infty = b_0, b_1, \dots, b_\ell = a$  such that each pair  $b_{i-1}, b_i$  is collinear. Hence  $\rho := [\infty, b_1, \dots, b_{\ell-1}, a] \in \mathcal{L}_\infty(\mathcal{D})$ . Moreover, for each  $b \in \Omega$  and each  $\sigma = [a, a_1, \dots, a_{k-1}, b] \in \text{Mor}(a, b)$ , we also have  $\tau := [\infty, b_1, \dots, b_{\ell-1}, a, a_1, \dots, a_{k-1}, b] \in \mathcal{L}_\infty(\mathcal{D})$ , and  $\sigma = \rho^{-1} \cdot \tau$ . In particular, the category  $\mathcal{C}(\mathcal{D})$  is completely determined by the set  $\mathcal{L}_\infty(\mathcal{D})$ . Thus, just as the term *Mathieu groupoid* is applied in the literature to both  $\mathcal{C}(\text{PG}(2, 3))$  and  $\mathcal{L}_\infty(\text{PG}(2, 3))$ ,

so also the term *Conway groupoid* is used for both  $\mathcal{C}(\mathcal{D})$  and  $\mathcal{L}_\infty(\mathcal{D})$  (although we tend to focus on the latter). The following result, which follows from [18, Lemma 3.1], is relevant.

**LEMMA 2.1.** [18] *Let  $\mathcal{D}$  be a pliable 4-hypergraph for which each pair of points is collinear. Let  $\infty_1, \infty_2$  be points of  $\mathcal{D}$ . Then  $\pi_{\infty_1}(\mathcal{D}) \cong \pi_{\infty_2}(\mathcal{D})$  (as permutation groups).*

This lemma can be strengthened so that we obtain the same conclusion, supposing only that  $\mathcal{D}$  is a pliable, connected 4-hypergraph. The lemma allows us to talk about “the” hole stabilizer of such a hypergraph without having to specify the base point  $\infty$ . A similar statement also holds for the sets  $\mathcal{L}_{\infty_1}(\mathcal{D})$  and  $\mathcal{L}_{\infty_2}(\mathcal{D})$ , allowing us to talk about “the” hole stabilizer of a pliable, connected 4-hypergraph.

**2.1. Some examples.** For this section, we need some definitions: for positive integers,  $n, k, \lambda$ , a  $2 - (n, k, \lambda)$ -*design*  $(\Omega, \mathcal{B})$  consists of a set  $\Omega$  of “points” of size  $n$ , and a multiset  $\mathcal{B}$  of  $k$ -element subsets of  $\Omega$  (called “lines”) such that any 2-subset of  $\Omega$  lies in exactly  $\lambda$  lines. The design  $(\Omega, \mathcal{B})$  is called *simple* if there are no repeated lines (that is,  $\mathcal{B}$  is a set, rather than a multiset). If  $k = 4$  and  $(\Omega, \mathcal{B})$  is a simple  $2 - (n, 4, \lambda)$ -design, then  $(\Omega, \mathcal{B})$  is a connected 4-hypergraph. Further, if in addition  $(\Omega, \mathcal{B})$  is pliable, that is, if distinct lines intersect in a set of size at most 2, then  $(\Omega, \mathcal{B})$  is called *supersimple*.

The search for examples of interesting new Conway groupoids, which we report on, has focussed almost exclusively on the situation where the 4-hypergraph  $\mathcal{D} = (\Omega, \mathcal{B})$  is a supersimple  $2 - (n, 4, \lambda)$  design. In particular, Lemma 2.1 applies and the isomorphism class of the hole-stabilizer is, up to permutation isomorphism, independent of the choice of the point  $\infty$ .

Let us first consider a somewhat degenerate case: it turns out that  $\pi_\infty(\mathcal{D}) = \text{Alt}(\Omega \setminus \{\infty\})$  if and only if  $\mathcal{L}_\infty(\mathcal{D}) = \text{Alt}(\Omega)$ , and  $\pi_\infty(\mathcal{D}) = \text{Sym}(\Omega \setminus \{\infty\})$  if and only if  $\mathcal{L}_\infty(\mathcal{D}) = \text{Sym}(\Omega)$ . In these cases the puzzle-construction sheds no new light on the groups in question, and can be safely ignored. It turns out that for very many of the supersimple  $2 - (n, k, \lambda)$  designs  $\mathcal{D}$  examined, the corresponding Conway groupoid is of this type. Indeed it turns out that if  $n$  is sufficiently large relative to  $\lambda$ , then  $\mathcal{L}_\infty(\mathcal{D})$  always contains  $\text{Alt}(\Omega)$ .

**LEMMA 2.2.** [21, Theorem E(3)] *If  $\mathcal{D} = (\Omega, \mathcal{B})$  is a supersimple  $2 - (n, 4, \lambda)$  design with  $n > 144\lambda^2 + 120\lambda + 26$ , then  $\mathcal{L}_\infty(\mathcal{D}) \supseteq \text{Alt}(\Omega)$ .*

We shall have more to say about the relationship between  $n$  and  $\lambda$  in Theorem 4.1. However this crude bound is sufficient to show, for example, that the Conway groupoids for the point-line designs of projective spaces  $\text{PG}(r, 3)$  and affine spaces  $\text{AG}(r, 4)$  contain  $\text{Alt}(\Omega)$  whenever  $r \geq 5$ . The first infinite family of examples without this property was studied in [18].

**EXAMPLE 2.3.** The *Boolean quadruple system of order  $2^m$* , where  $m \geq 2$ , is the design  $\mathcal{D}^b = (\Omega^b, \mathcal{B}^b)$  such that  $\Omega^b$  is identified with the set of vectors in  $\mathbb{F}_2^m$ , and

$$\mathcal{B}^b := \{ \{v_1, v_2, v_3, v_4\} \mid v_i \in \Omega^b \text{ and } \sum_{i=1}^4 v_i = \mathbf{0} \}.$$

Equivalently, we can define

$$\mathcal{B}^b = \{v + W \mid v \in \Omega^b, W \leq \mathbb{F}_2^m, \dim(W) = 2\};$$

that is,  $\mathcal{B}^b$  is the set of all affine planes of  $\Omega^b$ . It is easy to see that  $\mathcal{D}$  is both a  $3-(2^m, 4, 1)$  Steiner quadruple system and a supersimple  $2-(2^m, 4, 2^{m-1} - 1)$  design. It turns out that  $\pi_\infty(\mathcal{D}^b)$  is trivial. In addition, the Conway groupoid  $\mathcal{L}_\infty(\mathcal{D}^b)$  is equal to the group of translations  $E_{2^m}$  acting transitively on  $\Omega^b$  [18, Theorem B, Section 5].

The approach in the literature to finding new examples from supersimple designs has tended to be organised in terms of the behaviour of the hole stabilizer (as a subgroup of  $\text{Sym}(\Omega \setminus \{\infty\})$ ). For a Boolean quadruple system  $\mathcal{D}^b$ , the trivial group  $\pi_\infty(\mathcal{D}^b)$  is clearly intransitive on the set  $\Omega \setminus \{\infty\}$ . However, examples with intransitive hole-stabilizers seem quite rare. We only know three other examples. They are given in [18, Table 1], and have parameters:

- (1)  $(n, \lambda) = (16, 6)$  and  $\pi_\infty(\mathcal{D}) \cong \text{Sym}(3) \times \text{Sym}(3) \times \text{Sym}(3) \times \text{Sym}(3) \times \text{Sym}(3)$ ;
- (2)  $(n, \lambda) = (17, 6)$  and  $\pi_\infty(\mathcal{D}) \cong \text{Sym}(8) \times \text{Sym}(8)$ ;
- (3)  $(n, \lambda) = (49, 18)$  and  $\pi_\infty(\mathcal{D}) \cong \text{Sym}(24) \times \text{Sym}(24)$ .

QUESTION 2.4. *Apart from Boolean quadruple systems, are there infinitely many supersimple designs for which the hole-stabilizers are intransitive?*

We next turn our attention to the situation where the hole-stabilizers are transitive on  $\Omega \setminus \{\infty\}$ . Examples for which this action is imprimitive also seem to be rare. Only one example appears in [18, Table 1]: it has parameters  $(n, \lambda) = (9, 3)$  and  $\pi_\infty(\mathcal{D}) \cong \text{Alt}(4) \wr C_2$ .

QUESTION 2.5. *Are there more examples of supersimple designs  $\mathcal{D}$  for which  $\pi_\infty(\mathcal{D})$  is transitive and imprimitive?*

In [21] two infinite families of designs are studied for which the hole-stabilizers are primitive. To describe them we need the following set-up: Let  $m \geq 2$  and  $V := (\mathbb{F}_2)^{2m}$  be a vector space equipped with the standard basis. Define

$$(2.1) \quad e := \begin{pmatrix} 0_m & I_m \\ 0_m & 0_m \end{pmatrix}, \quad f := \begin{pmatrix} 0_m & I_m \\ I_m & 0_m \end{pmatrix} = e + e^T,$$

where  $I_m$  and  $0_m$  represent the  $m \times m$  identity and zero matrices respectively. We write elements of  $V$  as row vectors and define  $\varphi(u, v)$  as the alternating bilinear form  $\varphi(u, v) := uf v^T$ . We also set  $\theta(u) := ueu^T \in \mathbb{F}_2$ , so that

$$\theta(u + v) + \theta(u) + \theta(v) = \varphi(u, v).$$

(Note that the right hand side equals  $uev^T + veu^T$  while the left hand side is  $u(e + e^T)v^T$ .) Finally, for each  $v \in V$  define  $\theta_v(u) := \theta(u) + \varphi(u, v)$ , and note that  $\theta_0 = \theta$ .

EXAMPLE 2.6. The *Symplectic quadruple system of order  $2^{2m}$* , where  $m \geq 2$ , is the design  $\mathcal{D}^a = (\Omega^a, \mathcal{B}^a)$ , where  $\Omega^a := V$  and

$$\mathcal{B}^a := \left\{ \{v_1, v_2, v_3, v_4\} \mid v_1, v_2, v_3, v_4 \in \Omega^a, \sum_{i=1}^4 v_i = \mathbf{0}, \sum_{i=1}^4 \theta(v_i) = 0 \right\}.$$

By [21, Theorem B] for  $m \geq 3$  and [18, Table 1] for  $m = 2$ ,  $\mathcal{L}_\infty(\mathcal{D}^a) \cong 2^{2m}.\text{Sp}_{2m}(2)$ , while  $\pi_\infty(\mathcal{D}^a) \cong \text{Sp}_{2m}(2)$ . Indeed, taking  $\infty$  to be the zero vector in  $V$ ,  $\pi_\infty(\mathcal{D}^a) = \text{Isom}(V, \varphi)$ , the isometry group of the formed space  $(V, \varphi)$ .



EXAMPLE 2.7. The *Quadratic quadruple systems of order  $2^{2m}$* , where  $m \geq 3$ , are the designs  $\mathcal{D}^\varepsilon = (\Omega^\varepsilon, \mathcal{B}^\varepsilon)$ , for  $\varepsilon \in \mathbb{F}_2$ , such that  $\Omega^\varepsilon := \{\theta_v \mid v \in V, \theta(v) = \varepsilon\}$  and

$$\mathcal{B}^\varepsilon := \left\{ \{\theta_{v_1}, \theta_{v_2}, \theta_{v_3}, \theta_{v_4}\} \mid \theta_{v_1}, \theta_{v_2}, \theta_{v_3}, \theta_{v_4} \in \Omega^\varepsilon, \sum_{i=1}^4 v_i = \mathbf{0} \right\}.$$

By [21, Theorem B],  $\mathcal{L}_\infty(\mathcal{D}^\varepsilon) \cong \mathrm{Sp}_{2m}(2)$ , the isometry group of  $\varphi$ , while  $\pi_\infty(\mathcal{D}^\varepsilon) \cong \mathrm{O}_{2m}^{\varepsilon'}(2)$ , where  $\varepsilon' = \pm$  and  $\varepsilon = (1 - \varepsilon'1)/2$  (as an integer in  $\{0, 1\}$ ).

We remark that the set of lines in  $\mathcal{D}^a$  coincides with the set of translates of the totally isotropic 2-subspaces of  $\Omega^a$ . This alternative interpretation provides a link with Example 2.3. The designs in Example 2.7 can be rephrased similarly (see [20, §6]). Note, too, that Example 2.7 can be extended to include the case  $m = 2$ , but only for  $\varepsilon = 0$ . In this case, [18, Table 1] asserts that  $\mathcal{L}_\infty(\mathcal{D}^0) \cong \mathrm{Sym}(6)$  and  $\pi_\infty(\mathcal{D}^0) \cong \mathrm{O}_4^+(2) = \mathrm{Sym}(3) \wr \mathrm{Sym}(2)$ .

The examples described thus far represent all those known for which  $\mathcal{D}$  is a supersimple design. Note that in Examples 2.3, 2.6 and 2.7, the Conway groupoid  $\mathcal{L}_\infty(\mathcal{D})$  is always a group – we shall have more to say on this phenomenon in §4.2.2.

### 3. Conway groupoids and codes

In this section we consider certain codes that arise naturally from the supersimple designs initially used to define Conway groupoids. We use the following terminology from coding theory. A *code* of length  $m$  over an alphabet  $Q$  of size  $q$  is a subset of vertices of the Hamming graph  $H(m, q)$ , which is the graph  $\Gamma$  with vertex set  $V(\Gamma)$  consisting of all  $m$ -tuples with entries from  $Q$ , and such that two vertices are adjacent if they differ in precisely one entry. Consequently, the (Hamming) distance  $d(\alpha, \beta)$  between two vertices  $\alpha, \beta \in V(\Gamma)$  is equal to the number of entries in which they differ. If  $Q$  is a finite field  $\mathbb{F}_q$ , then we identify  $V(\Gamma)$  with the space  $\mathbb{F}_q^m$  of  $m$ -dimensional row vectors. In this case a code is called *linear* if it is a subspace of  $\mathbb{F}_q^m$ . We only consider linear codes in this section.

The *support* of a vertex  $\alpha = (\alpha_1, \dots, \alpha_m) \in V(\Gamma)$  is the set  $\mathrm{supp}(\alpha) = \{i \mid \alpha_i \neq 0\}$ , and the *weight* of  $\alpha$  is  $\mathrm{wt}(\alpha) = |\mathrm{supp}(\alpha)|$ . Given a code  $C$  in  $H(m, q)$ , the *minimum distance* of  $C$  is the minimum of  $d(\alpha, \beta)$  for distinct codewords  $\alpha, \beta \in C$ , and for a vertex  $\beta \in V(\Gamma)$ , the distance from  $\beta$  to  $C$  is defined as

$$d(\beta, C) = \min\{d(\beta, \alpha) \mid \alpha \in C\}.$$

The *covering radius*  $\rho$  of  $C$  is the maximum of these distances:

$$\rho = \max\{d(\beta, C) \mid \beta \in V(\Gamma)\}.$$

For  $i = 0, \dots, \rho$  we let

$$C_i = \{\beta \mid d(\beta, C) = i\},$$

so  $C_0 = C$ , and we call the partition  $\{C, C_1, \dots, C_\rho\}$  of  $V(\Gamma)$  the *distance partition of  $C$* .

A code is *completely regular* if its distance partition is *equitable*, that is, if the number of vertices in  $C_j$  adjacent to a vertex in  $C_i$  depends only on  $i, j$  and not on the choice of the vertex (for all  $i, j$ ). Such codes have a high degree of combinatorial symmetry, and have been studied extensively (see, for example, [11, 16, 28] and more recently [7, 8, 10, 29, 30, 31]). Additionally, certain distance regular graphs can be described as coset graphs of completely regular codes [11, p.353], and so such

codes are also of interest to graph theorists. *Completely transitive codes*, which are a subfamily of completely regular codes with a high degree of algebraic symmetry, have also been studied (see [9, 22, 35] for example).

For linear codes, the *degree*  $s$  of a code is the number of values that occur as weights of non-zero codewords. The *dual degree*  $s^*$  of a linear code  $C$  is the degree of its dual code  $C^\perp$ , where  $C^\perp$  consists of all  $\beta \in V(\Gamma)$  such that the dot product  $\alpha \cdot \beta := \sum_i \alpha_i \beta_i$  is equal to zero, for all codewords  $\alpha \in C$ . The covering radius  $\rho$  of a code  $C$  is at most  $s^*$ , and  $\rho = s^*$  if and only if  $C$  is *uniformly packed (in the wide sense)* [5]. Completely regular codes are necessarily uniformly packed [11], but only a few constructions of uniformly packed codes that are not completely regular are known [30].

REMARK 3.1. Various families of codes have been introduced in the literature bearing the name “uniformly packed”, and so to provide clarity for the reader, we give a brief description of these families. In [33], Semakov et al. introduced uniformly packed codes, which were subsequently generalised by Bassalygo et al. [4]. It is the codes defined in [4] that we call “uniformly packed (in the wide sense)”.

Goethals and Van Tilborg, feeling that the definition given in [4] was too general, introduced a family of codes in [23], which they also called uniformly packed. The Goethals and Van Tilborg codes are necessarily uniformly packed (in the wide sense). They are also necessarily completely regular, so the existence of uniformly packed codes (in the wide sense) that are not completely regular shows that the Goethals and Van Tilborg codes are a proper subset.

In this paper we do not limit ourselves to the Goethals and Van Tilborg codes; hence whenever we use the phrase “uniformly packed” from now on, we will be referring to codes that have the property defined in [4], that is to codes that are “uniformly packed (in the wide sense)”.

**3.1. The ternary Golay code.** For a hypergraph or design  $\mathcal{D} = (\Omega, \mathcal{B})$ , its *incidence matrix* is the matrix whose columns are indexed by the points of  $\Omega$ , whose rows are indexed by the lines of  $\mathcal{B}$ , and such that the  $(a, \ell)$ -entry is 1 if the point  $a$  lies in the line  $\ell$ , and is zero otherwise. The row vectors are therefore binary  $n$ -tuples, where  $n = |\Omega|$ , and we may interpret their entries as elements of any field. For a field  $F$  of order  $q$ , the code  $C_F(\mathcal{D})$  is defined as the linear span over  $F$  of the rows of the incidence matrix of  $\mathcal{D}$ . It is contained in the Hamming graph  $H(n, q)$ .

In [14], the authors considered the code  $C_{\mathbb{F}_3}(\text{PG}(2, 3))$ . They also constructed certain subcodes of this code, proving that one was the *ternary Golay code*. Note that the ternary Golay code which Conway et al. refer to is usually called the extended ternary Golay code in the coding theory literature. It is a ternary  $[12, 6, 6]$  code, which when punctured gives the perfect ternary  $[11, 6, 5]$  Golay code. We now describe their construction.

Let  $\mathcal{C} = C_{\mathbb{F}_3}(\text{PG}(2, 3))$ , and let  $p \in \mathcal{P}$ , the point set of  $\text{PG}(2, 3)$ . Conway et al. define

$$\mathcal{C}_p = \{\alpha \in \mathcal{C} \mid \alpha_p = -\sum_{i \in \mathcal{P}} \alpha_i\},$$

and prove that the restriction of  $\mathcal{C}_p$  to the coordinates  $\mathcal{P} \setminus \{p\}$  is isomorphic to the  $[12, 6, 6]_3$  ternary Golay code [14, Prop. 3.2], which has automorphism group  $M_{12}$ . It is this fact that is used by Conway et al. to prove that  $\pi_\infty(\text{PG}(2, 3)) \cong M_{12}$ , [14, Thm. 3.5].

We now show that the full code  $\mathcal{C}$  also has interesting properties which, to our knowledge, have not been observed previously.

**THEOREM 3.2.**  *$\mathcal{C}$  is a ternary  $[13, 7, 4]$  code that is uniformly packed (in the wide sense), but not completely regular.*

In order to prove this, we consider the following subcode of  $\mathcal{C}$ , which Conway et al. use to determine certain properties of  $\mathcal{C}$ :

$$\mathcal{C}' = \{\alpha \in \mathcal{C} \mid \sum \alpha_i = 0\}.$$

For a line  $\ell$  of  $\text{PG}(2, 3)$ , let  $h_\ell$  denote the weight 4 vector in  $H(13, 3)$  with  $i$ -entry equal to 1 if  $i \in \ell$ , and zero otherwise.

**LEMMA 3.3.** [14, Prop. 3.1] *Let  $\alpha \in \mathcal{C}$ . Then*

- i)  $\text{wt}(\alpha) \equiv 0 \text{ or } 1 \pmod{3}$ ;
- ii)  $\alpha \in \mathcal{C}'$  if and only if  $\text{wt}(\alpha) \equiv 0 \pmod{3}$ ;
- iii)  $\mathcal{C}^\perp = \mathcal{C}'$ ;
- iv)  $\mathcal{C}$  and  $\mathcal{C}'$  have minimum distance 4, 6, respectively, and the weight 4 codewords in  $\mathcal{C}$  are precisely the vectors  $\pm h_\ell$ , for lines  $\ell$  of  $\text{PG}(2, 3)$ ;
- v) Let  $\ell$  be a line in  $\text{PG}(2, 3)$ . Then

$$(3.1) \quad \sum_{i \in \mathcal{P}} \alpha_i = \sum_{i \in \ell} \alpha_i.$$

- vi)  $\dim(\mathcal{C}) = 7$  and  $\dim(\mathcal{C}') = 6$ .

We also need the following concepts. A vertex  $\beta$  is said to *cover* a vertex  $\alpha$  if  $\beta_i = \alpha_i$  for all  $i \in \text{supp}(\alpha)$ . So for example,  $(2, 1, 1, 0)$  covers  $(2, 1, 0, 0)$  in  $\mathbb{F}_3^4$ . A set  $\mathcal{S}$  of vertices of weight  $k$  in  $\mathbb{F}_q^m$  is a  $q$ -ary  $t - (m, k, \lambda)$  *design* if every vertex of weight  $t$  in  $\mathbb{F}_q^m$  is covered by exactly  $\lambda$  elements of  $\mathcal{S}$ . It is known that for a linear completely regular code  $\mathcal{C}$  in  $\mathbb{F}_q^m$  with minimum distance  $\delta$ , the set of codewords of weight  $k$  forms a  $q$ -ary  $\lfloor \frac{\delta}{2} \rfloor - (m, k, \lambda)$  design for some  $\lambda$  [36, Theorem 2.4.7].

**PROOF OF THEOREM 3.2.** The parameters of the code follow from Lemma 3.3, so let us first show that  $\mathcal{C}$  is uniformly packed. By Lemma 3.3, and since the codes have length 13, the possible weights of non-zero codewords of  $\mathcal{C}^\perp = \mathcal{C}'$  are 6, 9 and 12, and hence  $s^* \leq 3$ . Also, since  $\mathcal{C}$  has minimum distance 4, the covering radius  $\rho$  is at least 2, so  $2 \leq \rho \leq s^* \leq 3$ . Let  $\ell_i$  for  $i = 1, 2$  denote two of the four lines that contain the point 1 in  $\text{PG}(2, 3)$ , and let  $x, y$  be points on  $\ell_1, \ell_2$  respectively, that are distinct from 1. Let  $\nu$  be any vertex of weight 3 with  $\text{supp}(\nu) = \{1, x, y\}$ . Since  $\text{PG}(2, 3)$  is a projective plane, it follows that  $|\text{supp}(\nu) \cap \ell| \leq 2$  for all lines  $\ell$  of  $\text{PG}(2, 3)$ . Thus, Lemma 3.3(iv) implies that  $d(\nu, \alpha) \geq 3$  for all codewords  $\alpha$  of weight 4. Since all other codewords in  $\mathcal{C}$  have weight at least 6, it follows that  $d(\nu, \mathcal{C}) \geq 3$ , and hence  $\rho = s^* = 3$ . Thus  $\mathcal{C}$  is uniformly packed.

Suppose that  $\mathcal{C}$  is completely regular. Then the set of codewords of weight 4 forms a 3-ary  $2 - (13, 4, \lambda)$  design for some  $\lambda > 0$ . However, by Lemma 3.3(iv), every vertex of weight 2 with constant non-zero entries is covered by exactly one codeword of weight 4, whereas a vertex of weight 2 with distinct non-zero entries is not covered by any codeword of weight 4. This contradiction proves that  $\mathcal{C}$  is not completely regular.  $\square$

**3.2. Conway groupoids and completely regular codes.** As we have seen, the Conway groupoid  $M_{13}$  is interesting in several different ways. Its hole stabilizer  $\pi_\infty(\mathcal{D})$  is multiply transitive, and hence primitive; the perfect Golay code over  $\mathbb{F}_3$  can be constructed from  $C_{\mathbb{F}_3}(\text{PG}(2, 3))$ ; and, moreover, the code  $C_{\mathbb{F}_3}(\text{PG}(2, 3))$  has some interesting and rare properties. Thus it is natural to ask if one can construct other interesting codes from supersimple designs  $\mathcal{D}$  for which  $\pi_\infty(\mathcal{D})$  is acting primitively. This question was addressed in [21] for the designs defined in Examples 2.6 and 2.7.

**THEOREM 3.4.** *The following hold:*

- (a) *For  $\varepsilon \in \mathbb{F}_2$ ,  $C_{\mathbb{F}_2}(\mathcal{D}^\varepsilon)$  is a completely transitive code with covering radius 3 and minimum distance 4.*
- (b)  *$C_{\mathbb{F}_2}(\mathcal{D}^a)$  is a completely transitive code with covering radius 4 and minimum distance 4.*

The following question arises naturally as a consequence of Theorems 3.2 and 3.4.

**QUESTION 3.5.** *Let  $\mathcal{D} = (\Omega, \mathcal{B})$  be a supersimple  $2 - (n, 4, \lambda)$  design such that  $\pi_\infty(\mathcal{D})$  is a primitive subgroup  $\text{Sym}(\Omega \setminus \{\infty\})$  which does not contain  $\text{Alt}(\Omega \setminus \{\infty\})$ . Does there exist a prime  $r$  such that  $C_{\mathbb{F}_r}(\mathcal{D})$  is completely regular, or at the very least uniformly packed (in the wide sense)?*

We remarked earlier that for each point  $p \in \text{PG}(2, 3)$ , Conway et al define a code  $\mathcal{C}_p$  with the property that  $\text{Aut}(\mathcal{C}_p) \cong M_{12}$ . They use this code to show that  $\pi_\infty(\text{PG}(2, 3)) \cong M_{12}$ : namely, in [14, Proposition 3.3] they show that the elementary move  $[p, q]$  sends  $\mathcal{C}_p$  to  $\mathcal{C}_q$ , and from this they deduce in [14, Proposition 3.4] that  $\pi_\infty(\text{PG}(2, 3)) \leq M_{12}$ . Equality then follows by an explicit computation. Arguing in this spirit, with  $\mathcal{C}$  being one of the codes  $\mathcal{C}_{\mathbb{F}_2}(\mathcal{D})$  of Theorem 3.4, it is relatively straightforward to show that for any point  $p$  of  $\mathcal{D}$ , the code  $\mathcal{C}_p$  obtained by puncturing  $\mathcal{C}$  at  $p$  has automorphism group isomorphic to the stabilizer  $\text{Stab}_{\text{Aut}(\mathcal{D})}(p)$  of  $p$ . Moreover, the elementary move  $[p, q]$  sends  $\mathcal{C}_p$  to  $\mathcal{C}_q$ , and we deduce that  $\pi_\infty(\mathcal{D}) \leq \text{Stab}_{\text{Aut}(\mathcal{D})}(p)$ . This fact can be used to give an alternative proof (to that given in [21]) of the isomorphism type of  $\mathcal{L}_\infty(\mathcal{D})$  for the designs  $\mathcal{D}$  defined in Examples 2.6 and 2.7.

#### 4. Classification results

The programme to classify Conway groupoids has, thus far, been restricted to the situation where  $\mathcal{D}$  is a supersimple  $2 - (n, 4, \lambda)$  design, a family which includes  $\text{PG}(2, 3)$ . In this section we describe the progress that has been made in this setting.

**4.1. Relation between  $n$  and  $\lambda$ .** In this subsection we connect the relative values of the parameters  $\lambda$  and  $n$  with the behaviour of the hole stabilizer  $\pi_\infty(\mathcal{D})$  in its action on  $\Omega \setminus \{\infty\}$ . By examining the examples given in §2.1 one may be lead to observe the following: if we fix  $\lambda$  and allow  $n$  to increase, the way  $\pi_\infty(\mathcal{D})$  acts on  $\Omega \setminus \{\infty\}$  seems to move through the following states:

$$\text{trivial} \longrightarrow \text{intransitive} \longrightarrow \begin{matrix} \text{transitive} \\ \text{imprimitive} \end{matrix} \longrightarrow \text{primitive} \longrightarrow \begin{matrix} \text{Alt}(\Omega \setminus \{\infty\}) \text{ or} \\ \text{Sym}(\Omega \setminus \{\infty\}). \end{matrix}$$

This observation was proved and quantified in [18, 21]. We note first that two points in a  $2 - (n, 4, \lambda)$  design lie together on  $\lambda$  lines, and the set theoretic union of

these lines in a supersimple design has size  $2\lambda + 2$ . Thus for supersimple designs we must have  $n \geq 2\lambda + 2$ . On the other hand, Lemma 2.2 gives an upper bound in terms of  $\lambda$  beyond which, for all designs  $\mathcal{D}$ ,  $\pi_\infty(\mathcal{D})$  is  $\text{Alt}(\Omega \setminus \{\infty\})$  or  $\text{Sym}(\Omega \setminus \{\infty\})$ . This bound is refined in the result quoted below, and we make some comments about the proof in Remark 4.2.

**THEOREM 4.1.** ([18, Theorem B] and [21, Theorem E]) *Suppose that  $\mathcal{D}$  is a supersimple  $2 - (n, 4, \lambda)$  design, and  $\infty$  is a point of  $\mathcal{D}$ .*

- (1) *If  $n > 2\lambda + 2$ , then  $\pi_\infty(\mathcal{D})$  is non-trivial;*
- (2) *if  $n > 4\lambda + 1$ , then  $\pi_\infty(\mathcal{D})$  is transitive;*
- (3) *if  $n > 9\lambda + 1$ , then  $\pi_\infty(\mathcal{D})$  is primitive;*
- (4) *if  $n > 9\lambda^2 - 12\lambda + 5$ , then either  $\pi_\infty(\mathcal{D}) \supseteq \text{Alt}(\Omega \setminus \{\infty\})$ , or else  $\mathcal{D} = \text{PG}(2, 3)$ ,  $\pi_\infty(\mathcal{D}) = M_{12}$  (and  $\lambda = 1$ ).*

**REMARK 4.2.** (a) The proofs of the first three parts of Theorem 4.1 are independent of the Classification of the Finite Simple Groups (CFSG) (as is that of Lemma 2.2), but this is not true for part (4).

(b) Part (1) can be strengthened: in [18, Theorem B] it was shown that  $\pi_\infty(\mathcal{D})$  is trivial if and only if  $\mathcal{D}$  is a Boolean quadruple system, that is, one of the designs from Example 2.3.

(c) While the proofs of parts (2) and (3) are relatively straightforward counting arguments, the proof of Lemma 2.2 lies somewhat deeper. It relies on a lower bound proved by Babai [3] for the minimum number of points moved by a non-identity element of a primitive permutation group that does not contain the full alternating group. Babai's bound is combined with the observation that a move sequence  $[\infty, a, b, \infty]$  will have support of size at most  $6\lambda + 2$ ; now one must check that there exists such an element that is non-trivial, and the result follows.

(d) Part (4) (which is an improvement on Lemma 2.2) is obtained via the same method except that only those move sequences  $[\infty, a, b, \infty]$  for which  $a, b$  and  $\infty$  are collinear are considered, and the result of Babai is replaced by a stronger result due to Liebeck and Saxl [25]; it is here that the dependence on CFSG enters.

(e) It is natural to ask whether the bounds in Lemma 2.2 and Theorem 4.1 are best possible. Certainly part (1) cannot be improved, but for the others it is less clear.

**QUESTION 4.3.** *Can the quadratic function in Lemma 2.2 be replaced by a linear function?*

An immediate corollary of Theorem 4.1 is the following.

**COROLLARY 4.4.** *For a positive integer  $\lambda$ , there are only finitely many supersimple  $2 - (n, 4, \lambda)$  designs  $(\Omega, \mathcal{B})$  for which  $\mathcal{L}(\mathcal{D})$  does not contain  $\text{Alt}(\Omega)$ .*

This corollary suggests that a full classification for a given  $\lambda$  may be possible. This has been achieved for  $\lambda \leq 2$  in [18, Theorem C], but all other cases are open.

**THEOREM 4.5.** [18, Theorem C] *Let  $\mathcal{D} = (\Omega, \mathcal{B})$  be a supersimple  $2 - (n, 4, \lambda)$  design for which  $\mathcal{L}(\mathcal{D})$  does not contain  $\text{Alt}(\Omega)$ , and such that  $\lambda \leq 2$ . Let  $\infty \in \Omega$ . Then either*

- (1)  $\lambda = 1$ ,  $\mathcal{D} = \text{PG}(2, 3)$  and  $\pi_\infty(\mathcal{D}) = M_{12}$ ; or
- (2)  $\lambda = 2$ ,  $\mathcal{D}$  is the unique supersimple  $2 - (10, 4, 2)$  design and  $\pi_\infty(\mathcal{D}) = \text{Sym}(3) \wr \text{Sym}(2)$ .

The design in Theorem 4.5 (2) is connected to the family of designs in Example 2.7 (recall that  $\text{Sym}(3) \wr \text{Sym}(2) \cong O_4^+(2)$  and see the remark after Example 2.7). Theorem 4.5 pre-dates Theorem 4.1, but its proof is of a similar flavour. In this case, the result of Babai mentioned in Remark 4.2 is replaced by classical work of Manning classifying primitive permutation groups that contain non-identity elements moving less than 9 points, [26, 27].

QUESTION 4.6. *Can those  $2 - (n, 4, \lambda)$  designs  $\mathcal{D}$  be classified for which  $\mathcal{L}(\mathcal{D})$  does not contain  $\text{Alt}(\Omega)$  and  $\lambda$  is, say, 3, 4 or 5?*

Some remarks concerning a classification for  $\lambda = 3$  can be found in [21, §7.3].

**4.2. Extra structure.** In this section we consider two instances where we have been able to give a complete classification of Conway groupoids  $\mathcal{L}_\infty(\mathcal{D})$  subject to some set of conditions on the elementary moves, for supersimple designs  $\mathcal{D}$ .

4.2.1. *Collinear triples yielding trivial move sequences.* Here we consider [21, Theorem D] which was a critical ingredient in the proof of Theorem 4.1 (4) above, and which generalizes the classification of Conway groupoids associated with  $2 - (n, 4, 1)$  designs given in Theorem 4.5.

THEOREM 4.7. *Suppose that  $\mathcal{D}$  is a supersimple  $2 - (n, 4, \lambda)$  design, and that  $[\infty, a, b, \infty] = 1$  whenever  $\infty$  is collinear with  $\{a, b\}$ . Then one of the following is true:*

- (1)  $\mathcal{D}$  is a Boolean quadruple system and  $\pi_\infty(\mathcal{D})$  is trivial;
- (2)  $\mathcal{D} = \text{PG}(2, 3)$  (the projective plane of order 3) and  $\pi_\infty(\mathcal{D}) \cong M_{12}$ ; or
- (3)  $\pi_\infty(\mathcal{D}) \supseteq \text{Alt}(\Omega \setminus \{\infty\})$ .

Recall that a Boolean quadruple system was defined in Example 2.3. The proof of Theorem 4.7 given in [21] involves an interesting intermediate result, [21, Proposition 6.4]. This result asserts that any design  $\mathcal{D}$  which satisfies the hypotheses of Theorem 4.7 can be constructed in a rather curious way: one starts with a  $2 - (n, 2^{\alpha+1}, 1)$  design  $\mathcal{D}_0$  (for some  $\alpha \in \mathbb{Z}^+$ ) and one “replaces” each line in  $\mathcal{D}_0$  with a Boolean quadruple system of order  $2^{\alpha+1}$ . One thereby obtains a  $2 - (n, 4, 2^\alpha - 1)$  design satisfying the given hypothesis, and all such designs arise in this way.

4.2.2. *Regular two-graphs.* In this section we study three properties which turn out to be connected in the context of Conway groupoids.

Firstly, a  $2 - (n, 3, \mu)$  design  $(\Omega, \mathcal{C})$  is a *regular two-graph* if, for any 4-subset  $X$  of  $\Omega$ , either 0, 2 or 4 of the 3-subsets of  $X$  lie in  $\mathcal{C}$ . We are interested in those  $2 - (n, 4, \lambda)$  designs  $\mathcal{D}$  for which the pair  $(\Omega, \mathcal{C})$  is a regular two-graph, where  $\mathcal{C}$  is the set of triples of collinear points.

Secondly, we consider designs  $\mathcal{D} = (\Omega, \mathcal{B})$  that satisfy the following property:

$$(\triangle) \quad \text{if } B_1, B_2 \in \mathcal{B} \text{ such that } |B_1 \cap B_2| = 2, \text{ then } B_1 \triangle B_2 \in \mathcal{B}$$

where  $B_1 \triangle B_2$  denotes the *symmetric difference* of  $B_1$  and  $B_2$ .

Finally, we are interested in those designs for which  $\mathcal{L}_\infty(\mathcal{D})$  is a group. The following result which is (part of) [20, Theorems A and 4.2] connects these three properties. It is proved combinatorially.

THEOREM 4.8. *Let  $\mathcal{D} = (\Omega, \mathcal{B})$  be a supersimple  $2 - (n, 4, \lambda)$  design with  $n > 2\lambda + 2$ . Let  $\mathcal{C}$  denote the set of collinear triples of points in  $\Omega$ , and let  $\infty \in \Omega$ . Then the following hold.*

- (a) *If  $\mathcal{L}_\infty(\mathcal{D})$  is a group then  $\mathcal{L}_\infty(\mathcal{D})$  is primitive on  $\Omega$ .*

- (b) If  $(\Omega, \mathcal{C})$  is a regular two-graph then  $\pi_\infty(\mathcal{D})$  is transitive on  $\Omega \setminus \{\infty\}$ .
- (c) If  $(\Omega, \mathcal{C})$  is a regular two-graph and  $\mathcal{L}_\infty(\mathcal{D})$  is a group then  $\pi_\infty(\mathcal{D})$  is primitive on  $\Omega \setminus \{\infty\}$ .
- (d) If  $(\Omega, \mathcal{C})$  is a regular two-graph and  $\mathcal{D}$  satisfies  $(\Delta)$ , then  $\mathcal{L}_\infty(\mathcal{D})$  is a group.

It turns out that part (d) can be strengthened: in [20, Theorem 4.2]) we show that the group  $\mathcal{L}_\infty(\mathcal{D})$  is in fact a subgroup of automorphisms of  $\mathcal{D}$ , and is a 3-transposition group with respect to its set  $\mathcal{E}$  of elementary moves. This observation was combined in [20] with Fischer's classification of finite 3-transposition groups ([17]) to classify Conway groupoids arising from designs  $\mathcal{D}$  that satisfy the hypotheses of part (d).

The conditions of part (d) were also used in [20] in another way: it turns out that, for any point  $\infty \in \Omega$ , the assumptions of part (d), together with the condition  $n > 2\lambda + 2$ , imply that  $(\Omega \setminus \{\infty\}, \mathcal{C}_\infty)$  is a polar space in the sense of Buekenhout and Shult, where  $\mathcal{C}_\infty$  is the set of all triples of points in  $\Omega \setminus \{\infty\}$  which occur in a line with  $\infty$ . In fact, the polar space  $(\Omega \setminus \{\infty\}, \mathcal{C}_\infty)$  has the extra property that all lines in the space contain exactly three points. Such polar spaces were characterized in a special case by Shult [34] and then later, in full generality, by Seidel [32]. Seidel's result was used to derive the following classification result. This result provides an alternative proof for the classification of the associated Conway groupoids, avoiding the use of 3-transposition groups.

**THEOREM 4.9.** [20, Theorem C] *Let  $\mathcal{D} = (\Omega, \mathcal{B})$  be a supersimple  $2 - (n, 4, \lambda)$  design that satisfies  $(\Delta)$  and for which  $(\Omega, \mathcal{C})$  is a regular two-graph where  $\mathcal{C}$  is the set of collinear triples of points in  $\Omega$ . Then one of the following holds:*

- (a)  $\mathcal{D}$  is a Boolean quadruple system, as in Example 2.3;
- (b)  $\mathcal{D}$  is a Symplectic quadruple system, as in Example 2.6;
- (c)  $\mathcal{D}$  is a Quadratic quadruple system, as in Example 2.7.

Note that the structures of the corresponding hole stabilizers and Conway groupoids are listed in the relevant examples. One naturally wonders if this theorem can be strengthened:

**QUESTION 4.10.** *Can Theorem 4.9 be extended to cover the situation where  $(\Delta)$  does not hold? Are there any additional examples?*

We conclude by noting that the statements of Theorems 4.1 and 4.7 both require particular clauses to deal with  $M_{13}$ : it seems that, in the world of Conway groupoids,  $M_{13}$  is rather special. The following question connects this notion to the study of Conway groupoids with extra structure.

**QUESTION 4.11.** *Is  $M_{13}$  the only Conway groupoid which is not itself a subgroup of  $\text{Sym}(\Omega)$ , and for which the associated hole stabilizer  $\pi_\infty(\Omega)$  is a primitive subgroup of  $\text{Sym}(\Omega \setminus \{\infty\})$ ?*

Note that the Conway groupoids arising in Examples 2.6 and 2.7 have primitive hole stabilizers, but are subgroups of  $\text{Sym}(\Omega)$ .

## 5. Generation games

We have seen that a supersimple  $2 - (n, 4, \lambda)$  design provides a convenient structure by which to associate with each pair  $\{a, b\}$  of points a permutation  $[a, b]$

sending  $a$  to  $b$ . We conclude this survey by considering a few other combinatorial structures which might be exploited to find interesting new Conway groupoids.

**5.1. Working with triples.** For a  $2 - (n, 3, \mu)$  design  $(\Omega, \mathcal{C})$ , a map  $[\cdot, \cdot] : \Omega \times \Omega \rightarrow \text{Sym}(\Omega)$  is said to be a *pliable function associated with*  $(\Omega, \mathcal{C})$  if the following hold:

- (a) for each  $a, b \in \Omega$ ,  $[a, b]$  sends  $a$  to  $b$  and  $[a, b]^{-1} = [b, a]$ ;
- (b) for  $a \neq b$ ,  $\text{supp}([a, b]) = \{a, b\} \cup \{c \mid c \text{ is collinear with } a, b\}$ .

Here  $\text{supp}(g)$  (for  $g \in \text{Sym}(\Omega)$ ) means the set of points of  $\Omega$  moved by  $g$ , and a point  $c$  is *collinear* with  $\{a, b\}$  if  $\{a, b, c\} \in \mathcal{C}$ . We usually assume also that  $[a, a] = 1$  for all  $a \in \Omega$ .

For such a function, and for each  $a_0, a_1, \dots, a_k \in \Omega$ , define:

$$[a_0, a_1, a_2, \dots, a_k] := [a_0, a_1][a_1, a_2] \cdots [a_{k-1}, a_k],$$

to be a *move sequence* and for each  $\infty \in \Omega$ , define:

$$(5.1) \quad \mathcal{L}_\infty([\cdot, \cdot]) := \{[\infty, a_1, a_2, \dots, a_k] \mid k \in \mathbb{Z}, a_1, \dots, a_k \in \Omega\} \subseteq \text{Sym}(\Omega); \text{ and}$$

$$(5.2) \quad \pi_\infty([\cdot, \cdot]) := \{[\infty, a_1, a_2, \dots, a_{k-1}, \infty] \mid k \in \mathbb{Z}, a_1, \dots, a_{k-1} \in \Omega\} \subseteq \text{Sym}(\Omega \setminus \{\infty\})$$

to be the *Conway groupoid* and *hole-stabilizer*, respectively, associated with  $\infty$ . We have the following examples.

- (a) A supersimple  $2 - (n, 4, \lambda)$  design  $\mathcal{D} = (\Omega, \mathcal{B})$  determines a  $2 - (n, 3, 2\lambda)$  design  $(\Omega, \mathcal{C})$ , where  $\mathcal{C}$  is the set of collinear triples of  $\mathcal{D}$ . The elementary moves associated with  $\mathcal{D}$  determine a pliable function  $[\cdot, \cdot] : \Omega \times \Omega \rightarrow \text{Sym}(\Omega)$  associated with  $(\Omega, \mathcal{C})$ . Moreover  $\mathcal{L}_\infty(\mathcal{D}) = \mathcal{L}_\infty([\cdot, \cdot])$  and  $\pi_\infty(\mathcal{D}) = \pi_\infty([\cdot, \cdot])$ , for each  $\infty \in \Omega$ .
- (b) Any finite group  $G$  determines a pliable function  $[\cdot, \cdot] : G \times G \rightarrow \text{Sym}(G)$  associated with  $(G, \mathcal{C})$ , where  $\mathcal{C}$  is the set of all 3-subsets of  $G$ , by taking  $[a, b]$  to be right multiplication by  $a^{-1}b$ . Thus  $[a, b]$  is the unique element of the right regular action of  $G$  on  $G$  which maps  $a$  to  $b$ . Here  $(\Omega, \mathcal{C})$  is a  $2 - (n, 3, n - 2)$  design, where  $n = |G|$ , and  $\mathcal{L}_\infty([\cdot, \cdot]) \cong G$ ,  $\pi_\infty([\cdot, \cdot]) = 1$ . Observe that by (a) the Boolean  $2 - (2^m, 4, 2^{m-1} - 1)$  designs of Example 2.3 determine pliable functions of this type where  $G \cong (C_2)^m$ .
- (c) For an example which is not of either of these types consider the unique  $2 - (6, 3, 2)$  design  $(\Omega, \mathcal{C})$  whose lines are given by:

$$\begin{array}{ccccc} 012 & 023 & 034 & 045 & 051 \\ 124 & 235 & 341 & 452 & 513 \end{array}$$

Thus  $(\Omega, \mathcal{C})$  is the (extended) Paley two-graph with automorphism group  $\text{PSL}(2, 5)$ . For  $a, b \in \Omega$ , let  $[a, b] := \text{Id}_\Omega$  if  $a = b$  and otherwise set  $[a, b] := (a, b)(c, d)$  where  $\{a, b, c\}$  and  $\{a, b, d\}$  are the two lines containing  $\{a, b\}$  in  $\mathcal{C}$ . Then  $[\cdot, \cdot]$  becomes a pliable function associated with  $(\Omega, \mathcal{C})$ , and it is easy to show that  $\mathcal{L}_\infty([\cdot, \cdot]) = \text{Aut}(\mathcal{C}) = \text{PSL}(2, 5)$ .

- (d) More exotic examples arise also. For example the Higman–Sims sporadic simple group  $HS$  has a 2-transitive action on a set  $\Omega$  of degree 176, and  $\Omega$  forms the point set of a  $2 - (176, 3, 162)$  design admitting  $HS$  as a group of automorphisms. Moreover the setwise stabiliser in  $HS$  of an unordered pair  $\{a, b\}$  of distinct points has a unique central involution  $z_{a,b}$ . These involutions form a conjugacy class of  $HS$  of size  $15400 = \binom{176}{2}$



and, furthermore, the map  $[\cdot, \cdot] : \Omega \times \Omega$  given by  $[a, b] = z_{a,b}$  is a pliable function, yielding  $\mathcal{L}_\infty([\cdot, \cdot]) = HS$ . We are grateful to Ben Fairbairn for informing us of this example.

Notice that  $\mathcal{L}_\infty([\cdot, \cdot])$  formed a subgroup of  $\text{Sym}(\Omega)$  in several of the above examples. Under this assumption, we can prove the following:

**THEOREM 5.1.** *Let  $[\cdot, \cdot]$  be a pliable function associated with a  $2 - (n, 3, \mu)$  design  $(\Omega, \mathcal{C})$ , where  $\mu > 4$ , and suppose that  $\mathcal{L}_\infty([\cdot, \cdot])$  is a group. If  $n > \frac{3}{2}\mu$ , then  $\mathcal{L}_\infty([\cdot, \cdot])$  is primitive on  $\Omega$ .*

**PROOF.** Suppose that  $n > \frac{3}{2}\mu$  and  $\mu \geq 4$ . Then  $n > \mu + \frac{1}{2}\mu \geq \mu + 2$ , and we note that for distinct  $a, b$ ,  $|\text{supp}([a, b])| = \mu + 2$ , by the definition of a pliable function. Suppose that  $G$  acts imprimitively on  $\Omega$  with  $m$  blocks of size  $k$ , where  $n = mk$  and  $m > 1, k > 1$ . First we observe that  $m \geq 3$ . This holds because, if  $m = 2$ , then for points  $a, b$  in different blocks of imprimitivity, the elementary move  $[a, b]$  must interchange the two blocks, and hence  $[a, b]$  must move every point, contradicting the fact that  $\text{supp}([a, b]) = \mu + 2 < n$ . Now let  $a, b$  be distinct points in the same block of imprimitivity  $\Delta$ , and let  $y$  be any point fixed by  $[a, b]$  (such a point exists since  $n > \mu + 2$ ). By part (b) of the definition of a pliable function, it follows that  $g := [a, y]$  fixes  $b$ , so  $g$  must fix  $\Delta$  setwise, and hence  $y = a^g \in \Delta$ . This shows that every point fixed by  $[a, b]$  lies in  $\Delta$ , or equivalently that  $\Omega \setminus \Delta \subseteq \text{supp}([a, b])$ . Thus

$$\mu + 2 = |\text{supp}([a, b])| \geq (m - 1)k + 2 \quad \text{and hence} \quad \mu \geq (m - 1) \cdot \frac{n}{m}.$$

Rearranging this yields

$$(5.3) \quad n \leq \frac{m}{(m - 1)} \cdot \mu \leq \frac{3}{2} \cdot \mu$$

and this contradiction completes the proof.  $\square$

The bound given in Theorem 5.1 is achieved by at least one design. To see this we construct a pliable function for a  $2 - (9, 3, 6)$  design  $(\Omega, \mathcal{C})$  with the property that  $\mathcal{L}_\infty([\cdot, \cdot])$  is transitive but imprimitive. Let  $\Omega := (\mathbb{F}_3)^2$  and let  $\mathcal{C}$  be the complement of an affine plane of order 3, that is,

$$\mathcal{C} := \{\{a, b, c\} \mid a, b, c \in \Omega, a + b + c \neq 0\}.$$

For not necessarily distinct points  $a, b \in \Omega$ , set

$$[a, b] := \prod_{w+a+b \neq 0} (w, a + b - w).$$

Then  $[a, b]$  is an involution with support of size eight and a unique fixed point  $w = -a - b$ . For each  $\infty \in \Omega$ ,  $\mathcal{L}_\infty([\cdot, \cdot]) \cong (C_3 \times C_3) : C_2$  with the nine non-trivial involutions given by  $\{[a, b] \mid a, b \in \Omega\}$  (notice that  $[a, b] = [c, d]$  whenever  $a + b = c + d$ ). Furthermore, it is easy to see that  $\mathcal{L}_\infty([\cdot, \cdot])$  preserves a system of imprimitivity with three blocks of size 3.

In fact this example is just one of an infinite family of  $2 - (3^k, 3, 3^k - 3)$  designs constructed from complements of affine spaces with the property that they admit pliable functions with Conway groupoid an imprimitive group [19]. Just as the Boolean quadruple systems in Example 2.3 provided the “smallest” examples of designs satisfying the hypotheses of Theorems 4.7 and 4.9, one might hope that these  $2 - (3^k, 3, 3^k - 3)$  designs could play a similar role in this more general context. For this reason, we ask the following:

**QUESTION 5.2.** Let  $[\cdot, \cdot]$  be a pliable function associated with a  $2 - (n, 3, \mu)$  design  $(\Omega, \mathcal{C})$  and suppose that  $\mathcal{L}_\infty([\cdot, \cdot])$  is a group. If  $n > \mu + 3$ , is  $\mathcal{L}_\infty([\cdot, \cdot])$  primitive?

**5.2. Using 4-hypergraphs.** As discussed in Section 2, most of the interesting Conway groupoids known arise from  $2 - (n, 4, \lambda)$  designs. We gave one alternative approach in Subsection 5.1 based on triple systems. Here we discuss briefly a few other possibilities involving 4-hypergraphs which are not 2-designs. The following infinite family of examples was presented in [18, Example 4.1].

**EXAMPLE 5.3.** Let  $n \geq 3$ , let  $\Omega$  be a set of size  $2n$  consisting of the points  $\{x_i, y_i \mid 1 \leq i \leq n\}$ , and let  $\mathcal{B}$  be the set

$$\mathcal{B} := \{ \{x_i, y_i, x_j, y_j\} \mid 1 \leq i < j \leq n \}.$$

Then  $\mathcal{D} := (\Omega, \mathcal{B})$  is a connected, pliable 4-hypergraph, and, for any  $\infty \in \Omega$ , the Conway groupoid  $\mathcal{L}_\infty(\mathcal{D})$  and hole stabilizer  $\pi_\infty(\mathcal{D})$  are defined as in the first part of Section 2. It was noted in [18, Example 4.1] that  $\pi_\infty(\mathcal{D}) \cong \text{Sym}(2) \wr \text{Sym}(n-1)$  if  $n$  is odd, and that  $\pi_\infty(\mathcal{D})$  is an index 2 subgroup of  $\text{Sym}(2) \wr \text{Sym}(n-1)$  if  $n$  is even. We give a short proof of this assertion, and also show that  $G := \mathcal{L}_\infty(\mathcal{D})$  is a group, equal to  $\text{Sym}(2) \wr \text{Sym}(n)$  if  $n$  is odd, and to its index 2 subgroup  $(\text{Sym}(2) \wr \text{Sym}(n)) \cap \text{Alt}(n)$  if  $n$  is even.

The elementary moves are: for distinct  $i, j \in \{1, \dots, n\}$ ,

$$(5.4) \quad [x_i, x_j] = [y_i, y_j] = (x_i, x_j)(y_i, y_j) \text{ and } [x_i, y_j] = [y_i, x_j] = (x_i, y_j)(x_j, y_i),$$

together with the fixed point free involution  $[x_1, y_1] = \dots = [x_n, y_n] = \prod_{i=1}^n (x_i, y_i)$ . Since  $\mathcal{D}$  is connected, we may assume that  $\infty := x_1$ . It is readily checked that for each triple of elements  $a, b, c \in \Omega$  we have

$$[a, b]^{[b, c]} = [a^{[b, c]}, c].$$

Hence an argument in [20, Lemma 2.7] shows that  $G$  is a group. Moreover [20, Lemma 2.6] implies that  $G_0 := \pi_\infty(\mathcal{D}) = \text{stab}_G(\infty)$ . (Note that, although both of the cited results in [20] are stated and proved for supersimple designs, in fact the argument carries through for connected, pliable 4-hypergraphs.)

Next we see that  $G$  leaves invariant the system of imprimitivity  $\Delta$  given by

$$\Delta := \{ \{x_i, y_i\} \mid 1 \leq i \leq n \}.$$

Hence  $G_0$  fixes the block  $\{x_1, y_1\}$  and we have

$$G \leq \text{Sym}(2) \wr \text{Sym}(n) \text{ and } G_0 \leq \text{Sym}(2) \wr \text{Sym}(n-1).$$

Since  $G_0 = \text{stab}_G(\infty)$ ,  $G_0$  contains the elementary moves  $[x_i, x_j], [x_i, y_j]$  given in (5.4), for each  $i, j$  such that  $2 \leq i < j \leq n$ , and moreover,  $G_0$  contains the product of these two elements which is  $(x_i, y_i)(x_j, y_j)$ . Thus  $G_0$  induces  $\text{Sym}(n-1)$  on  $\{ \{x_i, y_i\} \mid i = 2, \dots, n \}$ . Indeed  $G_0 \cap \text{Alt}(2n-2)$  induces  $\text{Sym}(n-1)$ .

Now let  $K \cong \text{Sym}(2)^n$  denote the base group of the wreath product. Then, for distinct  $i, j$ ,  $K$  contains  $[x_i, x_j][x_i, y_j] = (x_i, y_i)(x_j, y_j)$ , and it follows that  $G \cap K$  contains all the even permutations in  $K$ . Together these points imply first that  $G_0$  contains

$$(\text{Sym}(2) \wr \text{Sym}(n-1)) \cap \text{Alt}(2n-2).$$

In particular,  $G_0$  has index at most 2 in  $\text{Sym}(2) \wr \text{Sym}(n-1)$ . If  $n$  is even then every product of elementary moves is an even permutation, so that  $G_0$  is as claimed. If  $n$  is odd, then  $G_0$  also contains

$$g := [x_1, x_2, y_1, x_1] = [x_1, x_2][x_2, y_1][y_1, x_1] = (x_1, y_1)(x_2, y_2)[y_1, x_1]$$

which is an odd permutation since  $[y_1, x_1] = \prod_{i=1}^n (x_i, y_i)$  is an odd permutation. Thus in this case  $G_0$  is the full wreath product  $\text{Sym}(2) \wr \text{Sym}(n-1)$ . Very similar arguments confirm the claims about  $G$  (note that  $|G| = |\Omega| |G_0|$ ).

The 4-hypergraphs in this family are not 2-designs since, for example, the pair  $\{x_1, x_2\}$  lies in a unique line, while  $\{x_1, y_1\}$  lies in  $n-1$  lines. On the other hand, every pair of points is contained in at least one line. Hypergraphs with this property are said to be *collinearly complete*, (see [1]; their study goes back to work of D. G. Higman and J. E. McLaughlin in [24]).

QUESTION 5.4. *Are there other interesting families of Conway groupoids arising from collinearly complete 4-hypergraphs which are not 2-designs?*

The family of connected pliable 4-hypergraphs extends beyond those which are collinearly complete. It includes, for example, generalised quadrangles with 4 points on each line. There are only finitely many such geometries, and it is shown in [19] that for each of them the Conway groupoid is the full alternating group.

QUESTION 5.5. *Are there interesting Conway groupoids arising from connected pliable 4-hypergraphs which are not collinearly complete?*

**5.3.  $M_{24}$ .** In the previous two subsections we have started with different geometries, and sought to “play” analogues of Conway’s original “game” in order to obtain groups and/or groupoids.

What about if one works backwards, that is to say, one starts with a group and seeks to define a game on an appropriate geometry that generates it? As we described at the start of this paper, this was Conway’s original approach: he came to define his game after observing certain structural coincidences between the groups  $\text{PSL}_3(3)$  and  $M_{12}$ .

In fact this structural coincidence can precisely be described as a ‘3-local equivalence’ (in the sense that  $\text{PSL}_3(3)$  and  $M_{12}$  have isomorphic 3-fusion systems) and one immediately wonders whether there are other (pairs of)  $p$ -locally equivalent groups whose structure can be exploited in some similar fashion to give a “natural” generation game.

For example, might the group  $M_{24}$  be amenable to such an analysis, perhaps via some analogue of the dualized game (described in §1.2) played on an appropriate geometry? Might there exist a Conway groupoid  $M_{25}$  – or perhaps, as Conway himself mentioned after a lecture given by the third author – might there be an  $M_{26}$ ? In this direction, the 3-local equivalence between  $M_{24}$  and  $\text{PSL}_3(3) : 2$  is particularly suggestive (note that the latter can be realised as a group of permutations on 26 letters). In any case, a generation game for  $M_{24}$  would be immensely interesting and would naturally lead one to wonder about the other sporadic simple groups.

QUESTION 5.6. *Can the group  $M_{24}$  be generated in a natural way via a generation game on some finite geometry?*

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF SOUTH WALES, TREForest, CF37 1DL, U.K.

*E-mail address:* `nick.gill@southwales.ac.uk`

HEILBRONN INSTITUTE FOR MATHEMATICAL RESEARCH, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF BRISTOL, U.K.

*E-mail address:* `neil.gillespie@bristol.ac.uk`

CENTRE FOR THE MATHEMATICS OF SYMMETRY AND COMPUTATION, UNIVERSITY OF WESTERN AUSTRALIA, AUSTRALIA, ALSO AFFILIATED WITH KING ABDULAZIZ UNIVERSITY JEDDAH, SAUDI ARABIA

*E-mail address:* `cheryl.praeger@uwa.edu.au`

HEILBRONN INSTITUTE FOR MATHEMATICAL RESEARCH, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF BRISTOL, U.K.

*E-mail address:* `js13525@bristol.ac.uk`